Problem IV.E ... pendulum in the wind 12 points; průměr 9,13; řešilo 47 studentů

Measure the period of the torsion pendulum oscillations as a function of the length of the thread. Use at least two types of thread materials. Determine as accurately as possible all the relevant parameters on which the period depends. *FYKOS forgot an experiment.*

In the solution, we will first familiarize ourselves with the theory behind the experiment, provide a detailed description of the setup, present the obtained results, discuss them, and finally draw the most accurate conclusions.

Our measuring device will be a stopwatch on a mobile phone. For comparison, we will use two types of thread – a metal wire and a thin cable.

Theory

Similarly to the compression or stretching of solids, the torsion also creates stress in the material to counteract the deformation. Consider a cylindrical object (e.g., a metal wire) whose one base we begin to twist along the axis of symmetry. That causes the stress in the material mentioned above. In the first-order approximation, we can assume the stress is proportional to the angle of rotation. There will also be a material constant G, known as the shear modulus (modulus of rigidity), which has the unit Pa, and it can be shown that it is related to Young's modulus E.

Since we are talking about rotational motion, we will describe the dynamics of the oscillations using the torque. For a cylindrical hinge, we can find the formula for the torque M that arises between the bases of the cylinder, either in the literature or on the internet, to be

$$M = -D\varphi = -\frac{\pi G r^4}{2d}\varphi\,,$$

where r is the radius, d length of the hinge, and φ is the angle of rotation between the bases. The constant D is called the torsion constant, and its different value for the same weight and different hinges would imply a different period of oscillation, as we shall see later. The minus sign indicates that the torque acts against the deflection and tries to return the material to the initial position. This formula is only valid with sufficient accuracy for small angles. Therefore, we will not deflect the weight too far from the equilibrium position.

The resulting torque will, therefore, act on the weight, which will periodically rotate and decelerate. The relationship between the angular acceleration ε and the torque M is

$$M = J\varepsilon\,,$$

where J is the moment of inertia of the rotating object. In our case, we used a long, thin rod fixed at the centre, so its moment of inertia is

$$J = \frac{1}{12}ml^2\,,$$

where m is its mass and l is its length. The equation of motion

$$\frac{1}{12}ml^2\varepsilon = -\frac{\pi G r^4}{2d}\varphi$$

is analogous to the equation of motion for the linear harmonic oscillator and leads to the period of oscillation

$$T = 2\pi\sqrt{\frac{ml^2 d}{6\pi G r^4}}\,.$$

The problem statement asks about the dependence of the period on the length of the hinge. We now see that the period increases with the square root of d (so the exponent to which we raise the length of the hinge in the formula for the period is $k = 1/2$). It also increases with the increasing moment of inertia of the weight but decreases rapidly with the radius and the shear modulus.

Setup and execution of the experiment

We conducted the experiment with two types of hinges – a metal wire and a thin cable covered with rubber. Thin ropes or threads are not suitable for the experiment – they can have a complex internal structure, and our theoretical model may not be suitable, but mainly, the periods of oscillation may last for minutes, at which point the losses due to friction or air resistance are no longer negligible. Even our cable is unsuitable for this experiment, considering our theory, as it is composed of two materials – a metal inside and a rubber cover around it. On the other hand, the problem statement does not specify any requirements for the type of thread, and our theory might eventually be valid even for this type of hinge.

We use a wooden rod for the weight. We wrap the thread around the rope and make a knot as close to the body of the rod as possible. We suspend the other end of the thread on another rod, which is placed horizontally at a sufficient height above the ground.

We use a stopwatch on a mobile phone to measure the period, and to increase the accuracy, we measure several consecutive oscillations. That reduces the influence of the human reaction time. For measuring after the rod has settled in a stationary position, we lightly tap it to start it rotating and observe when it comes to a stop. Once we see the first movement back towards the equilibrium position, we start the stopwatch. Similarly, we stop the stopwatch after a few oscillations in the same direction.

For the metal wire, we measure the time of the first ten consecutive oscillations, with the longest hinge leading to a time of under a minute, which is experimentally acceptable. For the cable, the period of one oscillation at the longest hinge length is around half a minute, so we always measure just the time of two oscillations.

Results

Firstly, we give the measured values of the rod and wire parameters and then the numerical values of the oscillation periods.

Tab. 1: Important parameters of the items used.

	length of the rod	weight of the bar	radius of the wire	radius of the inner metal of the cable	total radius of the cable
	$\frac{l}{\mathrm{cm}}$	$\frac{m}{\mathrm{g}}$	$\frac{r_\mathrm{d}}{\mathrm{mm}}$	$\frac{r_\mathrm{k}}{\mathrm{mm}}$	$\frac{r_\mathrm{c}}{\mathrm{mm}}$
average	50.1	68	0.37	0.35	0.80
uncertainty	0.1	1	0.01	0.01	0.01

According to the manufacturer, the rod should be 50 cm long, which we also measured using a tailor's tape measure, so we estimated the uncertainty to be one millimeter. We measured the radii of the wires using a micrometer to achieve the highest accuracy possible. We used a kitchen scale to determine the weight of the rod, and since several measurements did not give a different value, we estimated the uncertainty to be the smallest increment on the display, so 1 g.

We now give the values of measured periods for both hinges with different lengths. The following table shows the periods after dividing the measured time by the number of oscillations.

Tab. 2: Measured periods of oscillation as a function of the length of the hinge.

metal wire		cable	
$\frac{d_\mathrm{d}}{\mathrm{cm}}$	$\frac{T_\mathrm{d}}{\mathrm{s}}$	$\frac{d_\mathrm{k}}{\mathrm{cm}}$	$\frac{T_\mathrm{k}}{\mathrm{s}}$
71.4	4.95	70.5	27.8
66.4	4.83	65.1	27.0
61.4	4.61	59.9	25.1
56.4	4.43	53.8	24.6
51.4	4.20	48.5	23.3
46.4	3.98	43.3	22.0
41.4	3.71	37.8	21.0
36.4	3.49	32.5	19.1
31.4	3.23	26.5	17.7
26.4	2.96	21.3	15.7
21.4	2.68	15.8	13.6
16.4	2.30	10.5	10.9
11.4	2.07	7.4	9.6
		5.0	7.9
		2.3	5.8

In the case of the metal wire, we have given the values to hundredths of a second because the uncertainty of the measurement is spread evenly over the uncertainties of the individual period measurements due to the high number of periods in a single measurement. For the case of the cable, the values are given to the tens of a second because the number of periods in one measurement was significantly lower.

The dependence of the period of oscillations on the length of the hinge was plotted in the graphs in figure 1. Indeed, we see the period increasing with the increasing length of the hinge in the form of a square root. To verify the data trend, we fit it with a power dependence. The other, but equivalent, option is to logarithmize the lengths of the hinge and the periods and fit it with a line. If we use the formula from the theory, we should get

$$\ln(T) = \ln\left(2\pi\sqrt{\frac{ml^2}{6\pi G r^4}}\sqrt{d}\right) = \ln\left(2\pi\sqrt{\frac{ml^2}{6\pi G r^4}}\right) + \frac{1}{2}\ln(d)\ ,$$

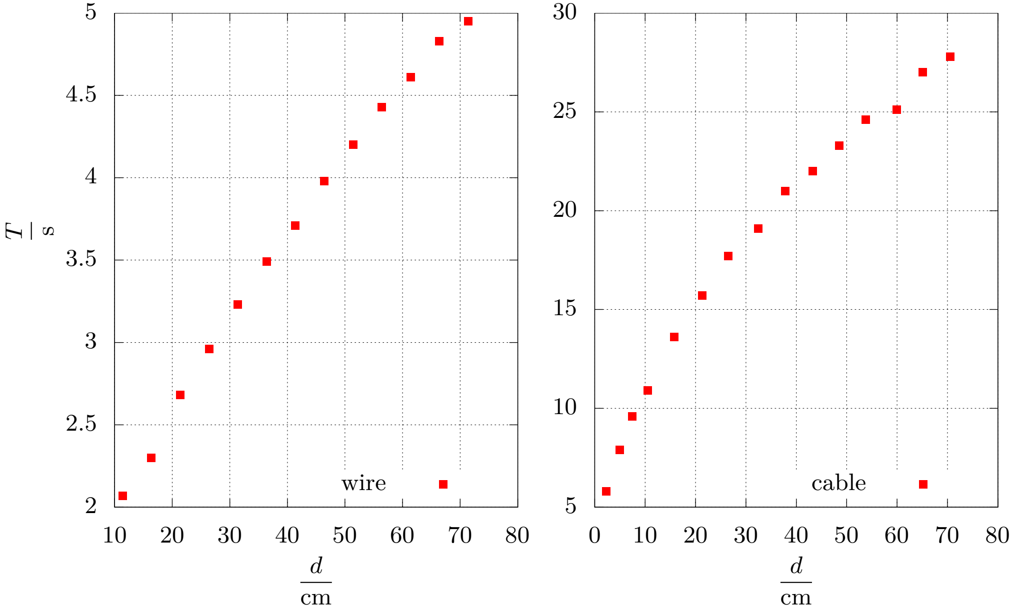

Fig. 1: Dependence of the period of the rod oscillations on the hinge length for both cables. Note the different axis ranges.

which, when plotted with values $y = \ln(T)$, $x = \ln(d)$, and $A = \ln\left(2\pi\sqrt{\frac{ml^2}{6\pi G r^4}}\right)$, gives the equation of a line

$$y = \frac{1}{2}x + A \;.$$

We expect a line with a slope of $1/2$ from the measured data. We will compare this value with the coefficient obtained from fitting the logarithmized data with a line, as shown in figure 2. Another way to visualize the data would be to plot the period as a function of the square root of the length of the hinge, where the dependence would be again linear.

The values found for the metal wire are $A_\mathrm{d} = (-0.53 \pm 0.03)$, $k_\mathrm{d} = (0.50 \pm 0.01)$, and for the cable $A_\mathrm{k} = (1.33 \pm 0.01)$, $k_\mathrm{k} = (0.46 \pm 0.01)$. From the coefficients k, it is evident that our theory is well applicable for the used hinges, as the expected value is $k = 1/2$.

For the metal wire, we use the value A_d to determine the shear modulus. We obtain it as

$$G_\mathrm{d} = \left(\frac{2\pi}{\exp(A_\mathrm{d})}\right)^2 \frac{ml^2}{6\pi r^4} = (5\,500 \pm 200)\mathrm{GPa}\,.$$

For the cable, such a calculation does not quite make sense since it is composed of multiple materials. However, we can still determine a constant that will govern the dependence of the period of oscillation on the material for a given length. The torsion constant contains no constitutive relations and only represents the proportionality between angular deflection and torque. However, as we have seen, the torsion constant depends on the length of the hinge. If

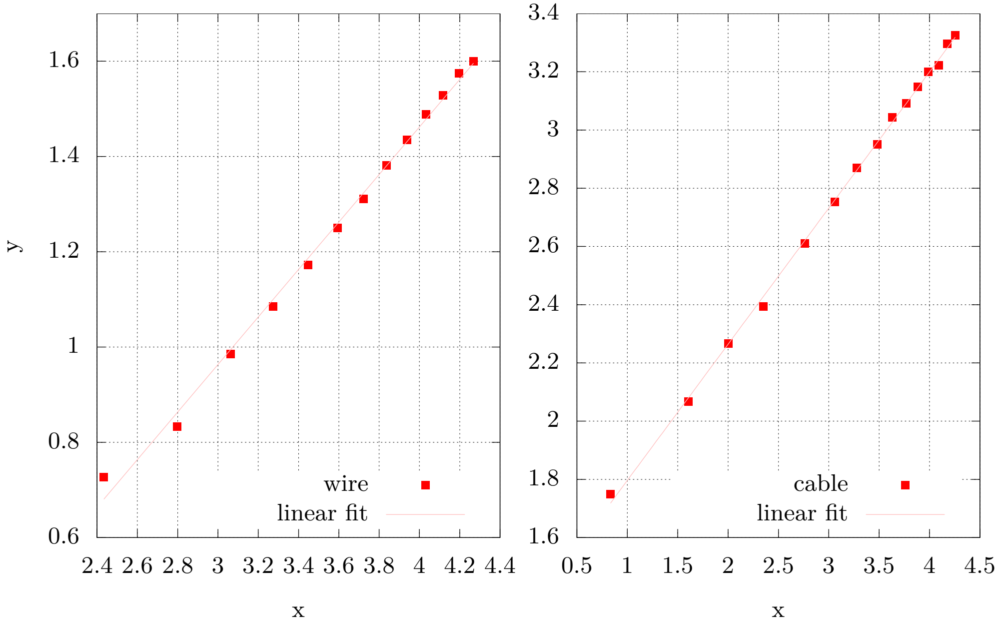

Fig. 2: Dependence of the logarithm of the period of oscillation of the rod on the logarithm of the length of the hinge for both hinges. Note the different axis ranges.

we multiply the two values together, we get a constant for the specific material of the hinge, which can be used to compare what the periods will be for two different threads with the same length of the hinge. Let us, therefore, introduce a constant $B = D \cdot d$ and calculate its value for both the cable and the metal wire.

For the cable, we get

$$B_{\mathrm{k}} = D_{\mathrm{k}} \cdot d = \frac{1}{12} m l^2 \left(\frac{2\pi}{\exp(A_{\mathrm{k}})} \right)^2 = (3.9 \pm 0.1) \cdot 10^{-3}\,\mathrm{N{\cdot}m{\cdot}rad^{-1}}\,,$$

whereas for the metal wire, we have

$$B_{\mathrm{d}} = \frac{1}{12} m l^2 \left(\frac{2\pi}{\exp(A_{\mathrm{d}})} \right)^2 = (1.61 \pm 0.02) \cdot 10^{-1}\,\mathrm{N{\cdot}m{\cdot}rad^{-1}}\,.$$

Discussion

From the data measured in logarithmic scales, it is evident that, indeed, we are dealing with a power-law relation with an exponent, as we assumed earlier. For the cable, the coefficient k_{k} turned out to be slightly lower than the expected value of 0.5, which could be due to the fact that the cable is composed of two materials, and our theory may not precisely apply to it. However, the deviation from the expected value is only a few percent, so we can consider our measurements to be satisfactory.

Similarly, it did not make sense to calculate the shear modulus for this hinge since it is composed of two materials. Thus, we only calculated the constant B, which is the product of the torsion constant of the thread and its length. This value was almost two orders of magnitude lower for the cable than for the metal wire, corresponding with significantly larger periods. Such a large difference lies in the different composition of the two hinges, where the cable contains considerably less metal than the wire, and the rubber insulation, from common experience, deforms much more easily.

Using the parameters of the used items, we have calculated the shear modulus for the metal wire as 5 500 GPa. That is absurdly high since common metals achieve values in the tens of GPa.[1] The wire thus exhibits greater resistance to torsion than what is solely dictated by its material. That might be because we have not used a perfectly cylindrical wire; rather, it was a piece of metal wire unwound from a larger quantity, so it had already undergone some changes in shape. These shape changes could have resulted in the wire not twisting uniformly along its entire length, as assumed in the theoretical part. However, the dependence on length is indeed square root, which is consistent with the theory.

Since the uncertainties of the results were in the order of a few percent, we can conclude that our theory, as well as the experimental setup, were consistent. Sources of the uncertainties include the resistance of the surrounding environment, which caused damping of the amplitude of deflection. If we considered the resistance proportional to angular velocity, we would get a dependence of the period on the magnitude of the resistive force (as in, e.g., damped harmonic oscillator) which would make the data analysis significantly more complicated. Therefore, we tried to work only in the range of small deflections so that angular velocities were not too high, allowing us to neglect this effect. Even our theory assumes only small deflections; for large deflections, the torque might not be a linear function of the amplitude of deflection. Additionally, we did not consider the moment of inertia of the hinge itself because, compared to its mass and dimensions, the moment of inertia is several orders of magnitude smaller.

Conclusion

We measured the dependence of the period of torsional oscillations of weights on the length of the hinge. For the metal wire, we found that the period has a square-root dependence on the length of the hinge, as our theory predicted. For the cable, we measured a slightly different dependence, which could have been caused by the presence of two different materials in the structure of the thread.

We determined the constant B (the product of the torsion constant of the thread with the length of the hinge) for the metal wire to be $D_\mathrm{k} = (3.9 \pm 0.1) \cdot 10^{-3}\,\mathrm{N{\cdot}m{\cdot}rad^{-1}}$, whereas for the cable is was significantly smaller, only $D_\mathrm{d} = (1.61 \pm 0.2) \cdot 10^{-1}\,\mathrm{N{\cdot}m{\cdot}rad^{-1}}$. We determined the

[1] `https://www.engineersedge.com/materials/shear_modulus_of_rigidity_13122.htm`

shear modulus of the metal wire to be $G_\mathrm{d} = (5\,500 \pm 200)\mathrm{GPa}$.

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