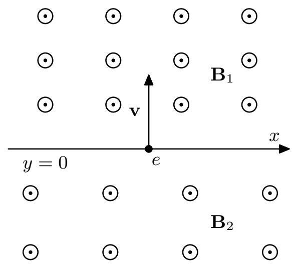
Problem IV.3 ... step here, step there

6 points; průměr 4,85; řešilo 62 studentů



Consider a homogeneous magnetic field of induction B_1, which spans a half-space bounded by the plane of interface $y = 0$, beyond which is an equally oriented, also homogeneous magnetic field of induction B_2. An electron flies out of the plane perpendicularly to it and the field lines (as in the figure) with velocity v. Determine the size and the direction of its average velocity parallel to the plane of the interface.

Bonus: Consider now that the magnitude of the field changes linearly as $B = B_0 (1 + \alpha y)$ and its direction is in the positive direction of the z-axis. Again, determine the magnitude and direction of the average velocity of the electron parallel to the interface plane. The electron is initially emitted as in the previous case. *Jarda is going one step forward and two steps back*

A magnetic force of magnitude $F = Bve$ acts on the electron. The force acts perpendicular to its velocity, so that the electron moves in a circle, with its radius derived from the centripetal force as

$$Bve = \frac{mv^2}{r} \quad \Rightarrow \quad r = \frac{mv}{Be} .$$

It is thus clear that the electron's radius of motion depends on the magnitude of the magnetic induction.

The time it takes the electron to circle half of the circle is

$$t = \frac{\pi r}{v} = \frac{\pi m}{Be} .$$

When the electron orbits half a circle, it enters a region with a different magnetic field, where the time and the radius are different. If the direction of the magnetic field lines in both half-spaces is the same (i.e., as in the problem statement), the electron bends in the same direction and returns to its starting point. Thus, in the time it takes for the direction of its velocity to rotate by 360 °, the position of the electron in the interface plane shifts by

$$2 (r_2 - r_1) = 2\frac{mv}{e} \left(\frac{1}{B_2} - \frac{1}{B_1}\right) .$$

If the sign of the magnetic inductions were opposite to each other, the electron would start to curve in the opposite direction at the interface, the direction of its velocity would not rotate by 360 °, and one of the radii would have a negative sign. However, nothing would change in the shape of the equations. We will not consider this configuration any further.

The electron acquires the initial velocity after time

$$t_1 + t_2 = \frac{\pi m}{e} \left(\frac{1}{B_1} + \frac{1}{B_2}\right) .$$

Its average speed in the interface plane is thus

$$v_\mathrm{a} = 2\frac{r_2 - r_1}{t_1 + t_2} = \frac{2v}{\pi} \frac{B_1 - B_2}{B_2 + B_1} .$$

Consider the direction of the x-axis as in the figure in the problem statement, the z-axis extending out of the figure towards the reader, and the y-axis pointing upwards. If the magnetic field is pointing in the direction of the z-axis as in the figure, and the initial velocity of the electron points in the positive y-axis direction, then for $B_1 > B_2$, the electron moves in the positive x-axis direction.

Bonus solution

The situation is the same as in the previous case, but now there is no more circular motion of the electron for $\alpha \neq 0$. Without loss of generality, we will assume that $\alpha > 0$. The equations of motion of the electron are

$$\ddot{x} = -\frac{e}{m} B_0 (1 + \alpha y) \dot{y} \,,$$
$$\ddot{y} = \frac{e}{m} B_0 (1 + \alpha y) \dot{x} \,,$$

where m is the mass of the electron, and the dots over the coordinates denote their time derivatives. To simplify notation, we introduce the substitution $eB_0/m = \gamma$. We note that the first of the equations can be integrated with respect to time to

$$\dot{x} = -\gamma \frac{1}{2\alpha} (1 + \alpha y)^2 + C \,,$$

where C is the integration constant. At time $t = 0$, the electron is in the $y = 0$ plane, and its x-component of the velocity is zero; therefore, $C = \gamma/(2\alpha)$.

Substituting $\dot{x}$ into the second equation of motion gives us a differential equation only in the variable y

$$\ddot{y} = \gamma (1 + \alpha y) \left(-\frac{\gamma}{2\alpha} (1 + \alpha y)^2 + \frac{\gamma}{2\alpha} \right) .$$

By multiplying by $\dot{y}$, we get

$$\ddot{y}\dot{y} = -\frac{\gamma^2}{2} \left(2y + 3\alpha y^2 + \alpha^2 y^3 \right) \dot{y} \,,$$

which we can integrate to obtain

$$(\dot{y})^2 = -\gamma^2 \left(y^2 + \alpha y^3 + \frac{1}{4}\alpha^2 y^4 + K \right) ,$$

where K is again an integration constant. From the initial condition, $K = -v^2/\gamma^2$. Since the left-hand side of the equation is a square, i.e., a non-negative number, the following must hold

$$y^2 + \alpha y^3 + \frac{1}{4}\alpha^2 y^4 - \frac{v^2}{\gamma^2} = \left(\left(1 + \frac{\alpha}{2} y \right) y + \frac{v}{\gamma} \right) \left(\left(1 + \frac{\alpha}{2} y \right) y - \frac{v}{\gamma} \right) \leq 0 \,.$$

Equalities arise for the solution of the equations

$$\alpha y^2 + 2y + \frac{2v}{\gamma} = 0 \quad \Rightarrow \quad y_2 = \frac{-1 - \sqrt{1 - \frac{2\alpha v}{\gamma}}}{\alpha}, y_3 = \frac{-1 + \sqrt{1 - \frac{2\alpha v}{\gamma}}}{\alpha} \,,$$
$$\alpha y^2 + 2y - \frac{2v}{\gamma} = 0 \quad \Rightarrow \quad y_1 = \frac{-1 - \sqrt{1 + \frac{2\alpha v}{\gamma}}}{\alpha}, y_4 = \frac{-1 + \sqrt{1 + \frac{2\alpha v}{\gamma}}}{\alpha} \,.$$

Thus, there are two real solutions for $2\alpha v/\gamma > 1$, three in the case of equality, and even four otherwise.

With no loss of generality, let us further assume that an electron from the $y = 0$ plane flies out with a velocity v in the direction of increasing y.

Let us first consider that all four roots exist. Then the inequality is satisfied on the intervals $y \in [y_1, y_2]$ and $y \in [y_3, y_4]$. However, the first interval does not satisfy the initial condition where $y = 0$, so we do not need to consider it any further. This is because the electron cannot get into this interval from the second one; it would need a higher initial speed to do so.

First, we describe qualitatively how the electron behaves in the interval $y \in [y_3, y_4]$. When launched perpendicular to $y = 0$, it has a maximum y-velocity v. However, the magnetic force bends its path to the point where its y-velocity drops to zero (if released in the direction of increasing y), which happens at $y = y_4$. The electron then bends back in the direction $y = 0$, along a trajectory symmetric to the one it took to get to $y = y_4$. It again passes through the $y = 0$ plane perpendicular to it at a velocity of v, but in the opposite direction from which it took off. It makes a similar arc in the $y < 0$ plane. Because in this case, $y_3 > -1/\alpha$, the electron does not get into a region where the orientation of the magnetic field is opposite, so the magnetic force only ever turns the electron in one direction. You can see a simulation of the trajectory of this case in Figure 1.

The time that the electron travels in the direction of increasing y is the same as when it travels in the opposite direction. Therefore, we can express the total time T for its velocity to return to its initial state as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{v^2 - \gamma^2 y^2 \left(1 + \frac{\alpha}{2} y\right)^2} \quad \Rightarrow \quad T = 2 \int_{y_3}^{y_4} \frac{\mathrm{d}y}{\sqrt{v^2 - \gamma^2 y^2 \left(1 + \frac{\alpha}{2} y\right)^2}} \,.$$

During this time, the electron moves along the x-axis by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = -\gamma y \left(1 + \frac{\alpha}{2} y\right) \quad \Rightarrow \quad X = 2 \int_{y_3}^{y_4} \frac{-\gamma y \left(1 + \frac{\alpha}{2} y\right)}{\sqrt{v^2 - \gamma^2 y^2 \left(1 + \frac{\alpha}{2} y\right)^2}} \,\mathrm{d}y \,.$$

Let us first solve the first integral. By introducing the substitution $Y = 2\alpha/y$ and then putting $c = 2\gamma/(\alpha v)$, we modify it to the expression

$$T = \frac{2c}{\gamma} \int_{Y_3}^{Y_4} \frac{\mathrm{d}Y}{\sqrt{1 - c^2 Y^2 (1 + Y)^2}} \,.$$

Substituting $cY(1 + Y) = \sin u$, we get $\mathrm{d}Y = \cos u \,\mathrm{d}u / (c(2Y + 1))$, and after substituting $2Y + + 1 = \sqrt{1 + (4/c) \sin u}$ then

$$T = \frac{2}{\gamma} \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}u}{\sqrt{1 + \frac{4}{c} \sin u}} \,.$$

We convert the integral for X to the form

$$X = -\frac{4}{\alpha} \int_{Y_3}^{Y_4} \frac{cY (1 + Y) \,\mathrm{d}Y}{\sqrt{1 - c^2 Y^2 (1 + Y)^2}} \,,$$

from which we similarly get

$$X = -\frac{2v}{\gamma} \int_{-\pi/2}^{\pi/2} \frac{\sin u}{\sqrt{1 + \frac{4}{c} \sin u}} \,\mathrm{d}u \,.$$

The values of the integrals now depend only on the parameter c. The velocity in the x-axis is thus

$$v_x = \frac{X}{T} = -v\frac{\int_{-\pi/2}^{\pi/2}\frac{\sin u}{\sqrt{1+\frac{4}{c}\sin u}}\,\mathrm{d}u}{\int_{-\pi/2}^{\pi/2}\frac{\mathrm{d}u}{\sqrt{1+\frac{4}{c}\sin u}}} = v\ f\left(\frac{2B_0e}{\alpha m v}\right)\,.$$

If there are only two roots of the equation for $\dot{y}$, then $y \in [y_1, y_4]$, and the lower integration limit changes from y_3 to y_1. This time, however, we cannot use substitution for $cY\,(Y+1)$ as we did in the previous text because, on a given interval, the mapping between functions is no longer injective – we can find multiple Y for one u.

However, we can notice that the function $1/\left(\sqrt{1-c^2Y^2\,(1+Y)^2}\right)$ is symmetric about the point $-1/2$. Indeed, by setting $Y = -1/2 \pm \xi$, we get

$$\begin{aligned}\frac{1}{\sqrt{1-c^2\left(-\frac{1}{2}-\xi\right)^2\left(1-\frac{1}{2}-\xi\right)^2}} &= \\ = \frac{1}{\sqrt{1-c^2\left(\frac{1}{2}+\xi\right)^2\left(\frac{1}{2}-\xi\right)^2}} &= \\ = \frac{1}{\sqrt{1-c^2\left(-\frac{1}{2}+\xi\right)^2\left(1-\frac{1}{2}+\xi\right)^2}}\,. &\end{aligned}$$

Thus, we just need to integrate from $1/2$ to Y_4 and multiply the result by two. On this interval, the substitution is already injective, so we can express the total time as

$$T = \frac{4}{\gamma}\int_{\theta}^{\pi/2}\frac{\mathrm{d}u}{\sqrt{1+\frac{4}{c}\sin u}}\,,$$

where $c\,(-1/2)\,(1-1/2) = -c/4 = \sin\theta$. Here, $c < 4$ (see the condition for the number of roots), so there is always a solution.

Similarly, we find the value of the shift along the x-axis as

$$X = -\frac{4v}{\gamma}\int_{\theta}^{\pi/2}\frac{\sin u}{\sqrt{1+\frac{4}{c}\sin u}}\,\mathrm{d}u\,,$$

and then if $2\alpha v/\gamma > 1$, we get the average velocity

$$v_x = \frac{X}{T} = -v\frac{\int_{\theta}^{\pi/2}\frac{\sin u}{\sqrt{1+\frac{4}{c}\sin u}}\,\mathrm{d}u}{\int_{\theta}^{\pi/2}\frac{\mathrm{d}u}{\sqrt{1+\frac{4}{c}\sin u}}} = v\ g\left(\frac{2B_0e}{\alpha m v}\right)\,.$$

How does the electron behave qualitatively in this case? After the emission, it does a similar arc as in the case of $c > 4$, but on the other half-plane, it enters a region where the magnetic field has the opposite sign. Therefore, the magnetic force starts to act in the other direction. The y-component of the electron's velocity again increases to its maximum value and circles the loop in the other direction. The rest of the motion is symmetrical, except that the electron returns to $y = 0$. Its trajectory thus forms a sort of unclosed figure eight. You can see one such trajectory in Figure 3.

The last one is the case where $2\alpha v/\gamma = 1$. The electron gets to the line $y = -1/\alpha$, and this with its velocity parallel to it. On this line, the intensity of the magnetic field is zero; there is no force acting on the electron, and thus, the electron moves in a uniform linear motion. Therefore, its average velocity in the x-direction is v. The trajectory is shown in Figure 2.

As the c increases, the speed decreases to zero (because a larger c can be made by decreasing velocity v or by decreasing gradient α, so the electron moves more in a circle).

The fact that we cannot express the velocity dependence using standard functions is not a physical problem – in Figure 4, this function is plotted. The direction of the positive velocity v_x points in the direction of increasing x.

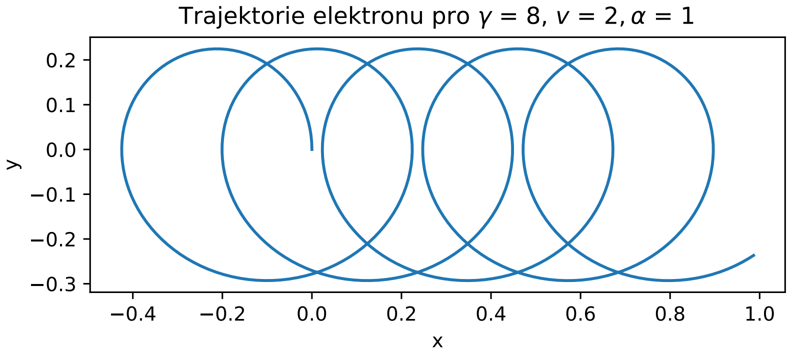


Fig. 1: Electron's trajectory for the parameters $\gamma = 8$, $v = 2$ a $\alpha = 1$ in SI units – corresponds to 4 roots.

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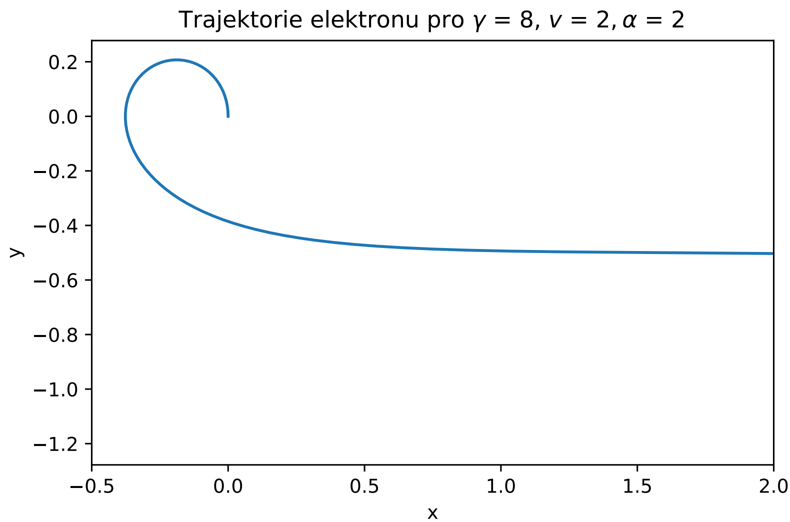

Fig. 2: Electron's trajectory for the parameters $\gamma = 8$, $v = 2$ a $\alpha = 2$ in SI units – corresponds to 3 roots.

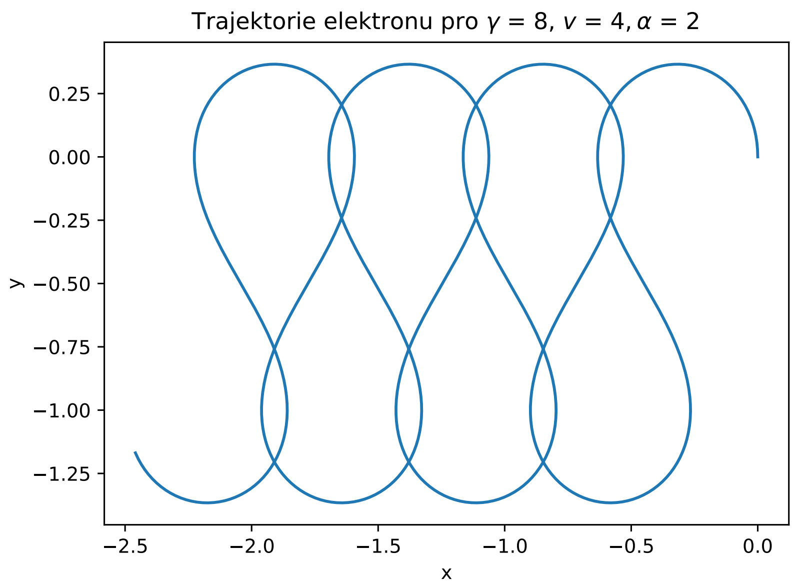

Fig. 3: Electron's trajectory for the parameters $\gamma = 8$, $v = 4$ a $\alpha = 2$ in SI units – corresponds to 2 roots.

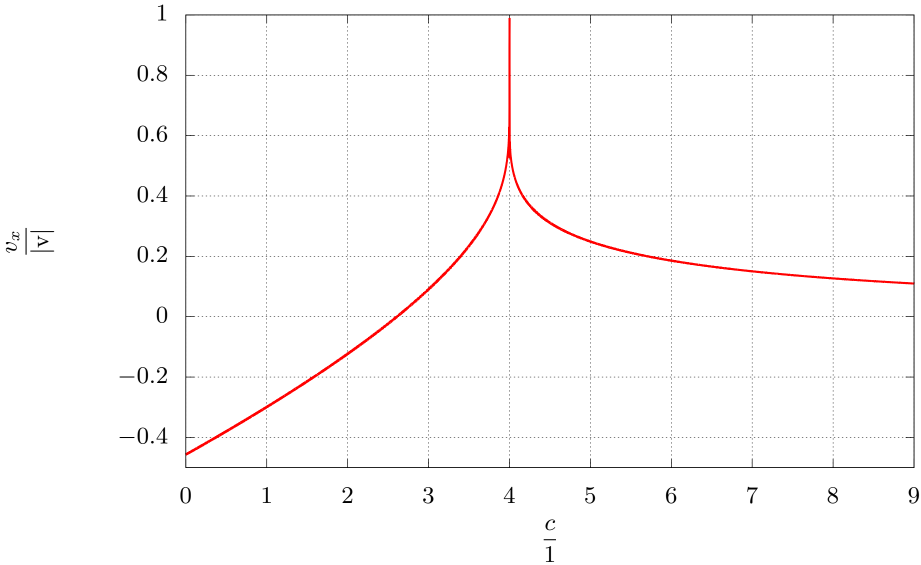

Fig. 4: Dependence of the x-component of velocity on the parameter c, expressed in the units of v.