Problem V.3 ... waiting for an elevator

6 points; průměr 4,43; řešilo 69 studentů

Karel uses an elevator in a building with a ground floor and 12 floors above it, while the height between floors is $h = 3.0\,\mathrm{m}$. Consider that the elevator accelerates half the time and decelerates half the time at a constant acceleration of $a = 1.0\,\mathrm{m{\cdot}s^{-2}}$ and that there is a 50% probability that the elevator is stationary on the ground floor. The rest of the probability is evenly distributed among the other floors. What is the expected waiting time for the elevator on each floor of the building? Neglect the time needed for opening doors.

Bonus Let us have 2 elevators in a twelve-story building. One elevator is always recalled to the ground floor. To which floor should we send the second one to minimize the average waiting time? Similarly, assume that half of the rides will start on the ground floor and the other half, with equally distributed probability, will start on any other floor.

Karel often waits for an elevator.

Let's start with the basic theory, which is quite simple because it is just an accelerated motion. The indexes i and j denote that the elevator goes from the i-th floor to the j-th floor. Half of the difference in height between floors $h_{i,j}$ will be covered by the elevator in half the total time $t_{i,j}$, which can be written as

$$\frac{h_{i,j}}{2} = \frac{1}{2}a\,\left(\frac{t_{i,j}}{2}\right)^2 .$$

From this equation, we express how long the elevator goes

$$4h_{i,j} = a\,t_{i,j}^2 \qquad \Rightarrow \qquad t_{i,j} = 2\sqrt{\frac{h_{i,j}}{a}} .$$

The height difference is a multiple of the absolute difference of the floors, so $h_{i,j} = h\,|j-i|$. We can modify the equation to the form

$$t_{i,j} = 2\sqrt{\frac{h}{a}}\sqrt{|j-i|} .$$

If we multiply this time by the probability p_i that the elevator will be on the i-th floor when we wait for it on the j-th floor, and we add all the probabilities, we get the total expected time to wait for the elevator on the jth floor

$$T_j = \sum_{i=0}^{12} p_i\sqrt{\frac{h}{a}}\sqrt{|j-i|} .$$

In the problem, the probabilities are given based on the assumption that people go equally to all floors and that they only get in and out of the building on the ground floor. Since people do not very often get out of windows, climb up eaves, etc., this is a reasonable assumption, at least as far as the ground floor is concerned. In actual operation, the assumption of evenness of going to each floor will not be satisfied because some people go out more often. But it is still the best estimation we can have without knowing the local conditions. The probabilities are $p_0 = 1/2$ and $p_i = 1/24$ for each $i \in \{1, 2, \ldots, 12\}$. The modification then gives

$$T_j = \sum_{i=0}^{12} p_i\sqrt{\frac{h}{a}}\sqrt{|j-i|} .$$

This is almost the required result, with the caveat that we have to calculate it. We can either type in the sum itself to have it computed by a program such as Wolfram Mathematica[1], or we can calculate the individual parts in Excel or Google Sheets, for example. We have decided to use a sample solution in Google Sheets and we publish the procedure under the link in the note.[2] The first sheet shows the calculated results of the basic problem. In region C5:O17 there are the running times for the case when the elevator is located on the floor listed in row 3 and the waiting person is on the floor listed in column A. In the region C20:O32, the times are already weighted by the probability that the elevator will be on that floor.

For individual floors we got results $T_0 = 4.2\,\mathrm{s}$, $T_1 = 5.5\,\mathrm{s}$, $T_2 = 5.8\,\mathrm{s}$, $T_3 = 6.1\,\mathrm{s}$, $T_4 = 6.4\,\mathrm{s}$, $T_5 = 6.7\,\mathrm{s}$, $T_6 = 7.0\,\mathrm{s}$, $T_7 = 7.4\,\mathrm{s}$, $T_8 = 7.7\,\mathrm{s}$, $T_9 = 8.1\,\mathrm{s}$, $T_{10} = 8.6\,\mathrm{s}$, $T_{11} = 9.1\,\mathrm{s}$, $T_{12} = 9.7\,\mathrm{s}$. If we wanted to call off the elevator to some floor so that it minimizes mean waiting time, then assuming that the elevator always makes it to that floor, it will be most efficient for it to go to the ground floor. This is a relatively expected result when half of the runs start on the ground floor.

If we wanted to make the problem more realistic, then we would have to consider that the elevator only accelerates for part of the time before reaching some maximum speed. Another problem would seem to be the opening and closing time of the door, but if the door is always closed, then our result will only change by a constant for any floor and not change the order of preference. The travel time will also affect our get-in and get-out times. In real traffic, especially on the ground floor, the longer you wait, the more people will be transported in the mean value, which will again delay you getting on, and much more if the elevator stops more than once on the way. Minimizing time also more or less goes against saving on utility bills.

Bonus

According to the assignment, we fix one elevator on the ground floor, and then for all combinations of the waiting person and the location of the second elevator we create a table where we again measure the waiting times. The procedure is shown on the second sheet of Google Sheets. The formulas are set up so that we again take the elevator travel times to a given floor, which we get from the formula for uniformly accelerated motion.

However, we always choose the one of the elevators that arrives first. In doing so, we try all positions of the second elevator (from the 1st to the 12th floor) and calculate a weighted average for all floors. We do not describe the intermediate results here, because the solution would then be unnecessarily lengthy. The result for our input parameters is that we can choose whether to place the second elevator on the 8th or 9th floor, since the mean waiting time is identical, namely 2.4 s. Compared to the situation with one elevator, people save 1.8 s, or 44 % of waiting time.

If you are wondering how it would end up if we decided to add a third elevator, the solution is not so simple, but we can expect that the second elevator should also change its location. We'll keep only the one on the ground floor and look at the total expected waiting times for different combinations of the 2nd and 3rd elevators. We can realize that a lot of combinations are duplicated or out of the question. However, we listed most of the possible ones and from these, we were able to determine that the optimal layout is the ground floor, 5th floor, and 10th

[1] We can also use the free WolframAlpha`https://www.wolframalpha.com/`.

[2] `https://docs.google.com/spreadsheets/d/10pQ3D7nYGmeDKZK2DYpOCqA2AjqMDX86f9s39_lGfn0`

floor. With this combination, the expected waiting time is just over 1.7 s. So we would save an additional 0.6 s per trip on average.

The task could be further varied. You can think of how unpleasantly complicated it would become if you placed 3 lifts, but you'd need 2 each time because you're always moving as a large group, and you can't fit in one lift. But in the real world, not all groups are the same size, the elevator doesn't always have time to arrive at the given floor, and our assumption of constant acceleration was also highly idealistic. It's going to make things even more complicated for buildings that don't have one significant floor but have, for example, underground parking.

A minor note might be that we were considering placing elevators only on integer floors but leaving the elevator on floor nine and three-quarters would be quite impractical from a technical point of view. We would probably still have concluded that it was convenient to have an elevator only on whole-number floors because at least those getting in on that floor don't have to wait. We do not deny that for some combinations of the number of lifts, number of floors, floor heights, and acceleration, it might be time-efficient to choose a non-integer floor for some lifts.

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